Reply to "Comment on 'Monostable array-enhanced stochastic resonance'"

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We take this opportunity to clarify issues raised in the preceding Comment by Soskin and McClintock [Phys. Rev. E. **66**, 013101 (2002)]. In particular, we provide further details and results to motivate and explicate the methodologies we have employed to investigate stochastic resonance in arrays of monostable elements.

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The preceding Comment by Soskin and McClintock [1] expresses confusion over some of the notation and interpretation of our demonstration [2] of multiple stochastic resonances in arrays of underdamped, monostable, nonlinear oscillators. We think the confusion arises mainly from the disparate methods and backgrounds our groups bring to the phenomenon of monostable stochastic resonance (SR). We hope that our comments here will clarify how our recent work [2] extends their original work [3] and how our methodology relates to theirs.

Soskin and McClintock agree that our main result is interesting and potentially important, but they are uncomfortable with our methodology, even in the case of a single element. The key issue is the definition and significance of our measure of the response of the oscillators. Our approach to *monostable* array-enhanced SR is conditioned by our prior work [4] with *bistable* array-enhanced SR. We consider SR to occur whenever a system's response—suitably defined—exhibits a local maximum as a function of noise. In our numerical experiments, we customarily quantify the response of a noisy, driven oscillator the same way we would in physical experiments. We generate a long time series $x[t]$ of the oscillator's position, take its fast Fourier transform $\tilde{x}[f]$, and create a power spectral density or spectrum $S[f]\propto|\tilde{x}[f]|^2$. We average many such spectra and estimate the background noise power by performing a local smooth fit $B[f]$ to the averaged spectrum near—but excluding—the drive frequency. Finally, we compute the ratio of the signal power at the drive frequency to the background noise power at the drive frequency

$$R=\frac{S[f_D]}{B[f_D]}, \tag{1}$$

which we conventionally express in decibels as $10\log_{10}R$ and refer to as the signal-to-noise ratio or SNR. [A better estimate of the SNR might involve integrating the area of the peak above the background, but experience has shown that Eq. (1) is sufficient for our purposes.] It is this definition that we employ in our discussion of noise-enhanced propagation in Sec. V and Fig. 7 of Ref. [2], which itself is an extension of our previous work [5].

As with physical experiments, our numerical experiments always assume a nonzero sampling time (or integration time step) dt and a finite *total* sampling time ΔT (per averaging segment). Because these times can affect the computed SNR, we are careful to keep them constant throughout any one series of experiments. We typically fix $dt=T_D/1024$ [2] and $\Delta T=128T_D$, where the drive period $T_D=1/f_D$. The former band limits our noise and the latter spoils the pure monochromaticity of our drive. We do not consider these to be serious limitations, because both white noise and monochromatic drives are nonphysical idealizations. Pure white noise would require an infinitesimal time step dt and an infinite band limit (or Nyquist frequency) $f_N=1/(2dt)$; a pure monochromatic drive, and its consequent diverging spectral spike, would require an infinite total sampling time ΔT and an infinitesimal frequency resolution $df=1/\Delta T$. Such conditions never obtained in practice and, contrary to one of the concerns of the Comment, our measure of response R never diverges.

We have adapted these numerical techniques to the case of monostable oscillators. Figure 1 displays the computed

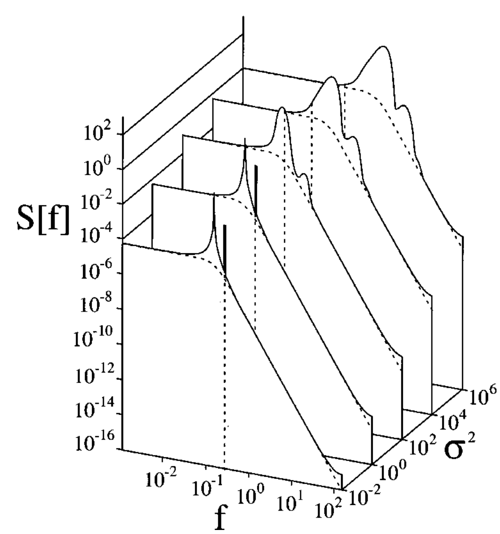

FIG. 1. The spectra $S[f]$ (solid curves) of a single, noisy, forced, damped monostable oscillator, for small and large noise variances, and for nonvanishing drive, consist of a large natural frequency peak and a narrow drive peak on smooth backgrounds $B[f]$ (dashed curves). Parameters from Ref. [2] are $m=1$, $\gamma=0.01$, $\alpha=1$, $\beta=0.1$, $A_D=0.1$, $f_D=0.285$.

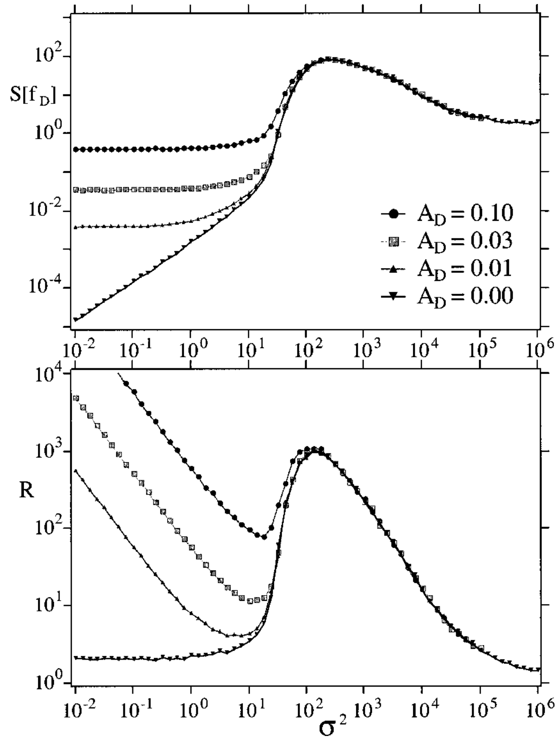

FIG. 2. Two measures of the response of the oscillator, $S[f_D]$ and $R=S[f_D]/B[f_D]$, for several different drive amplitudes, including zero drive amplitude. Both exhibit local maxima at nearly the same noise variances, but the R maxima are more pronounced. The maxima for sufficiently small drive amplitudes are indistinguishable from the maxima for vanishing drive.

spectra of a single monostable oscillator from Ref. [2], for both large and small noise variance σ^2, and for nonvanishing drive amplitude. Note how increasing noise variance causes the large natural frequency peak to shift to higher frequency and overwhelm the narrow drive peak (located by the vertical dashed lines). The spectra can be thought of as peaks riding on top of a smooth background (indicated by the curved dashed lines). As in the bistable case, we measure the system's response by the R of Eq. (1), but in this case, we estimate the background by performing a *global* smooth fit,

$$B[f]=B[f;B_0,f_{1/2}]=\frac{B_0}{1+(f/f_{1/2})^4}, \tag{2}$$

where we choose the two parameters B_0 and $f_{1/2}$ to match the spectra at low and high frequencies. This two-parameter fit, designed to be consistent with Eqs. (2) and (3) of Ref. [2], works well in practice for *all* of the noise variances we have considered, both small and large, as in Fig. 1.

This is a natural generalization of the technique we customarily use to estimate the noise background in the case of bistable SR. In that case, the spectrum typically consists of a narrow drive peak dominating a smooth background, thereby allowing a *local* background fit. However, in the monostable case, the natural frequency peaks often overwhelm the narrow drive peaks, thereby necessitating a *global* background fit.

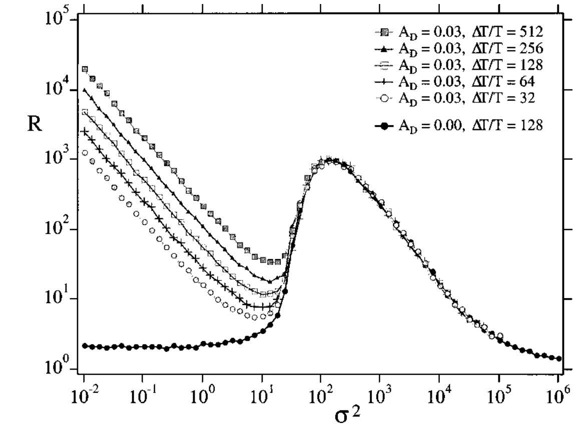

FIG. 3. Away from resonance, for small noise variance, the response R depends on the total sampling time ΔT. However, near resonance, R exhibits the same local maximum for a wide range of ΔT.

We emphasize that *our results are robust with respect to variations in the form of the background fit functions*. For example, in the monostable case, even the crude constant line fit $B[f]=B_0$ gives good results. (Division by the background removes the *rise* in the spectrum with noise variance, thereby focusing attention on the *shift* in the natural frequency peaks.)

Figure 2 displays both $S[f_D]$ and R as a function of noise variance σ^2, for a single oscillator, for vanishing and nonvanishing drive amplitudes. Both measures of response exhibit a local maximum as a function of noise as the natural frequency peak shifts over the drive frequency f_D. However, because the maximum in R is more pronounced than the maximum in $S[f_D]$, we employ R in studying the arrays. Note how both the position and height of the maximum are *independent* of the drive amplitude, at least for sufficiently small amplitudes.

We understand the essence of monostable SR to be the noise-induced matching of the system's natural frequencies

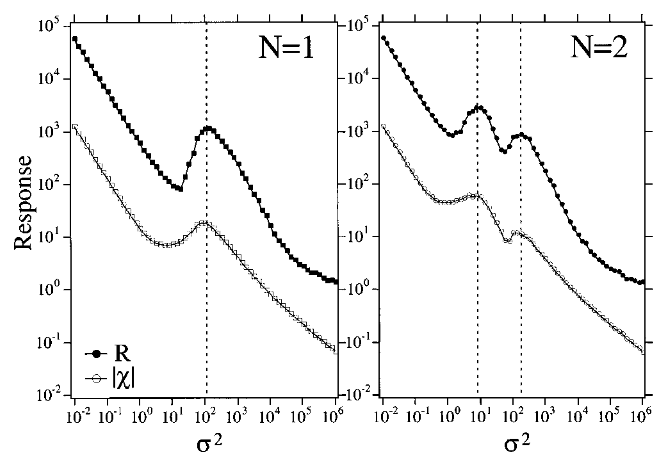

FIG. 4. Two measures of response, the susceptibility modulus $|\chi|$ and R, for both isolated $(N=1)$ and coupled $(N=2)$ oscillators. Both exhibit local maxima at the same noise variances, but the R maxima are more pronounced.

to a specific (drive) frequency. Since this frequency matching is independent of drive amplitude, for sufficiently small amplitudes, and for reasons of simplicity, in Ref. [2], we *present* data only for R at vanishing drive amplitude (except for Fig. 7, which illustrates the related phenomenon of noise-enhanced propagation, and is explicitly for nonvanishing drive). However, our research is based on an extensive series of numerical simulations, only a small subset of which was included in Ref. [2]. Much of these simulations, like those of Fig. 2, involve *nonvanishing* drive. They convince us that "We can calculate R with vanishing drive amplitude because R measures whether or not, and to what extent, a *natural frequency peak* is at the drive frequency, and because this frequency matching is the essential ingredient of monostable SR" [2]. Nevertheless, conceptually, we do consider R to be a measure of the response of the oscillator at typically small but nonvanishing drive amplitudes.

Figure 3 demonstrates the dependence of R on the total sampling time ΔT. Away from resonance, for small noise variance, R doubles every time ΔT doubles (and $df = 1/\Delta T$ halves). This is because the same finite drive power (mean square amplitude) is concentrated in a frequency bin half as wide. However, near the resonance, the natural frequency peak overwhelms the finite spike at the drive frequency, and R becomes independent of ΔT. Consequently, both the position and height of the local maximum (and hence the SR) are robust with respect to ΔT. In fact, for any large ΔT, we can find a drive amplitude A_D sufficiently small so that the system exhibits a SR.

In the Comment, Soskin and McClintock emphasize a different measure of response, the complex generalized susceptibility χ of Eq. (3) of Ref. [1] (which also appears in their original work [3]). Despite the fact that the integral expression for the real part of the susceptibility has a simple pole on the real axis, its numerical estimation is straightforward given our computed spectrum $S[f]$. When we do this, as in Fig. 4, we find that χ exhibits local maxima at the same noise variances as R. However, because R is simpler to compute, and because its local maxima are more pronounced, it is more useful to us. Note that since we are interested primarily in "where" things happen (so we can optimally tune the noise and coupling of the array), we do not need a measure of response that is in any technical sense "proportional" to the response.

The Comment also calls attention to a few misprints that escaped our proofreading. In the captions to Figs. 4 and 5, SNR should be replaced with R. (The relevant axes in these figures are correctly labeled R. We emphasize here that the original work [3] and our numerical estimation of the SNR of Eq. (5) of Ref. [1] confirm that, unlike R, this SNR decreases monotonically for these monostable oscillators.) An obvious factor of 2π is missing from the definitions of the modal frequencies f_0 and f_1 in Eqs. (2) and (3). Furthermore, the multiplier of κ in the definition of the antisymmetric frequency f_1 should indeed be 2 rather than 3. (In the antisymmetric mode, the equivalent springs between oscillators have nodes at their midpoints that effectively *halve* their lengths and *double* their spring constants.)

Finally, we agree with both of the Comment's closing remarks: Manifestations of SR in arrays of zero-dispersion oscillators may be quite impressive, and multiple SRs may be common in many higher dimensional nonlinear systems.

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